

Geometry of bisections of elliptic surfaces and Zariski N -plets II

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Abstract

In this article, we continue to study the geometry of bisections of certain rational elliptic surfaces. As an application, we give examples of Zariski $N + 1$ -plets of degree $2N + 4$ whose irreducible components are an irreducible quartic curve with 2 nodes and N smooth conics. Furthermore, by considering the case of $N = 2$ and combining with known results, a new Zariski 5-plet for reduced plane curves of degree 8 is given.

Introduction

In this article, we continue to study the topology of reducible plane curves via the geometry of elliptic surface as in [3, 6, 18, 20]. As for terminologies about Zariski pairs or Zariski N -plets, we refer to [2].

Let \mathcal{B} be a reduced plane curve with irreducible decomposition $\mathcal{B} = \mathcal{B}_1 + \cdots + \mathcal{B}_r$. A smooth conic \mathcal{C} is said to be a contact conic to \mathcal{B} if (i) no singular point of \mathcal{B} is contained in $\mathcal{B} \cap \mathcal{C}$ and (ii) for $\forall x \in \mathcal{B} \cap \mathcal{C}$, the intersection multiplicity $I_x(\mathcal{B}, \mathcal{C})$ at x is even.

For a smooth conic \mathcal{C} , let $f_{\mathcal{C}} : Z_{\mathcal{C}} \rightarrow \mathbb{P}^2$ be the double cover branched along \mathcal{C} . If \mathcal{C} is a contact conic to \mathcal{B} , for any irreducible component \mathcal{B}_i , $f_{\mathcal{C}}^* \mathcal{B}_i$ is either irreducible or of the form $\mathcal{B}_i^+ + \mathcal{B}_i^-$, where $\mathcal{B}_i^- = \sigma_{f_{\mathcal{C}}}^* \mathcal{B}_i^+$, $\sigma_{f_{\mathcal{C}}}$ being the covering transformation of $f_{\mathcal{C}}$. For the latter case, we say \mathcal{B}_i to be splitting or a splitting curve with respect to $f_{\mathcal{C}}$. As we have already seen in [3, 5, 17, 18], whether \mathcal{B}_i is splitting or not is a subtle and interesting question in the study of the topology of $\mathcal{C} + \mathcal{B}$. In fact, we have the following:

Proposition 0.1 ([17, Proposition 1.3]) *Let \mathcal{B} be an irreducible plane curve. Let \mathcal{C}_i ($i = 1, 2$) be contact conics to \mathcal{B} . If $f_{\mathcal{C}_1}^* \mathcal{B}$ is splitting and $f_{\mathcal{C}_2}^* \mathcal{B}$ is irreducible, then there exists no homeomorphism $h : (\mathbb{P}^2, \mathcal{C}_1 + \mathcal{B}) \rightarrow (\mathbb{P}^2, \mathcal{C}_2 + \mathcal{B})$ with $h(\mathcal{B}) = \mathcal{B}$.*

In [18], when \mathcal{B} is an irreducible plane quartic, a criterion for \mathcal{B} to be splitting with respect to $f_{\mathcal{C}}$ or not is given. We here recall it as follows:

Let \mathcal{Q} be a reduced quartic which is not a union of four concurrent lines and choose a general point z_o of \mathcal{Q} . We can associate a rational elliptic surface $S_{\mathcal{Q}, z_o}$ (see [6, 2.2.2], [20, Section 4]) to \mathcal{Q} and z_o , which is given as follows:

- (i) Let $f'_{\mathcal{Q}} : S'_{\mathcal{Q}} \rightarrow \mathbb{P}^2$ be a double cover branched along \mathcal{Q} .
- (ii) Let $\mu : S_{\mathcal{Q}} \rightarrow S'_{\mathcal{Q}}$ be the minimal resolution of $S'_{\mathcal{Q}}$.
- (iii) The pencil of curves passing through z_o on \mathbb{P}^2 gives rise to a pencil Λ_{z_o} of curves of genus 1 with a unique base point $(f'_{\mathcal{Q}} \circ \mu)^{-1}(z_o)$.

- (iv) Let $\nu_{z_o} : S_{\mathcal{Q}, z_o} \rightarrow S_{\mathcal{Q}}$ be the resolution of the indeterminacy for the rational map induced by Λ_{z_o} . We denote the induced morphism $\varphi_{\mathcal{Q}, z_o} : S_{\mathcal{Q}, z_o} \rightarrow \mathbb{P}^1$, which gives a minimal elliptic fibration. The map ν_{z_o} is a composition of two blowing-ups and the exceptional curve for the second blowing-up gives rise to a section O of $\varphi_{\mathcal{Q}, z_o}$. Note that we have the following diagram:

$$\begin{array}{ccccc}
S'_{\mathcal{Q}} & \xleftarrow{\mu} & S_{\mathcal{Q}} & \xleftarrow{\nu_{z_o}} & S_{\mathcal{Q}, z_o} \\
f'_{\mathcal{Q}} \downarrow & & \downarrow f_{\mathcal{Q}} & & \downarrow f_{\mathcal{Q}, z_o} \\
\mathbb{P}^2 & \xleftarrow{q} & \widehat{\mathbb{P}^2} & \xleftarrow{q_{z_o}} & (\widehat{\mathbb{P}^2})_{z_o},
\end{array}$$

where $f_{\mathcal{Q}, z_o}$ is a double cover induced by the quotient under the involution $[-1]_{\varphi_{\mathcal{Q}, z_o}}$ on $S_{\mathcal{Q}, z_o}$, which is given by the inversion with respect to the group law on the generic fiber. Furthermore q is a composition of a finite number of blowing-ups so that the branch locus becomes smooth and q_{z_o} is a composition of two blowing-ups.

In the following, we always assume that

a reduced quartic \mathcal{Q} is not a union of four concurrent lines.

By our assumption that \mathcal{C} is a contact conic, and since \mathcal{C} is rational, $(q \circ f_{\mathcal{Q}})^* \mathcal{C} = \mathcal{C}^+ + \mathcal{C}^-$. Furthermore, if we assume $z_o \in \mathcal{C}$, \mathcal{C}^{\pm} give rise to sections $s_{\mathcal{C}^{\pm}, z_o}$ of $\varphi_{\mathcal{Q}, z_o}$, respectively. Now we have

Theorem 0.1 [18, Theorem 2.1] *In the above setting, $f_{\mathcal{C}}^* \mathcal{Q}$ is splitting if and only if $s_{\mathcal{C}^+, z_o}$ is 2-divisible in $\text{MW}(S_{\mathcal{Q}, z_o})$*

Note that $s_{\mathcal{C}^-, z_o}$ is 2-divisible if and only if $s_{\mathcal{C}^+, z_o}$ is also two divisible, as $s_{\mathcal{C}^-, z_o} = [-1]s_{\mathcal{C}^+, z_o}$. Also our statement in Theorem 0.1 involves the choice of z_o , although the splitting property does not seem to depend on z_o . Later, we will resolve this discordance.

As we have seen in [18], when \mathcal{Q} is irreducible and has two nodes (2-nodal quartic) or one tacnode and a general point z_o , there exist contact conics $\mathcal{C}_1, \mathcal{C}_2$ through z_o such that (i) $f_{\mathcal{C}_1}^* \mathcal{Q}$ is splitting and (ii) $f_{\mathcal{C}_2}^* \mathcal{Q}$ is irreducible.

Since any homeomorphism $h : (\mathbb{P}^2, \mathcal{C}_1 + \mathcal{Q}) \rightarrow (\mathbb{P}^2, \mathcal{C}_2 + \mathcal{Q})$ satisfies $h(\mathcal{Q}) = \mathcal{Q}$, by Proposition 0.1, we have:

Proposition 0.2 *Let \mathcal{Q} be a 2-nodal quartic or an irreducible quartic with one tacnode. Let z_o be a general point and let \mathcal{C}_i ($i = 1, 2$) be as above. Then $(\mathbb{P}^2, \mathcal{C}_1 + \mathcal{Q})$ is not homeomorphic to $(\mathbb{P}^2, \mathcal{C}_2 + \mathcal{Q})$. In particular, if the combinatorial type of $\mathcal{C}_i + \mathcal{Q}$ ($i = 1, 2$) are the same, then $(\mathcal{C}_1 + \mathcal{Q}, \mathcal{C}_2 + \mathcal{Q})$ is a Zariski pair.*

Remark 0.1 In [18], we make use of existence/non-existence of dihedral covers of \mathbb{P}^2 branched along $\mathcal{C}_i + \mathcal{Q}$ ($i = 1, 2$) to prove Proposition 0.2.

In this article, we first consider a generalization of Proposition 0.2 as follows:

Let \mathcal{Q} be either a 2-nodal quartic or an irreducible quartic with one tacnode. Choose $N(N+1)$ contact conics $\mathcal{C}_j^{(i)}$ $i = 1, \dots, N+1$, $j = 1, \dots, N$ to \mathcal{Q} such that

- $f_{\mathcal{C}_j^{(i)}}^* \mathcal{Q}$ is splitting for $1 \leq j \leq N - i + 1$ and
- $f_{\mathcal{C}_j^{(i)}}^* \mathcal{Q}$ is irreducible if otherwise.

Put

$$\mathcal{B}^i := \mathcal{Q} + \sum_{j=1}^N \mathcal{C}_j^{(i)} \quad i = 1, \dots, N + 1.$$

Then we have

Theorem 0.2 *For i, k ($i \neq k$), there exists no homeomorphism $h : (\mathbb{P}^2, \mathcal{B}^{(i)}) \rightarrow (\mathbb{P}^2, \mathcal{B}^{(k)})$. In particular, if the combinatorial types of $\mathcal{B}^{(i)}$ ($i = 1, \dots, N + 1$) are all the same, $(\mathcal{B}^{(1)}, \dots, \mathcal{B}^{(N)})$ is a Zariski $(N + 1)$ -plet.*

In order to show that $\mathcal{C}_j^{(i)}$ ($i = 1, \dots, N + 1, j = 1, \dots, N$) such that $\mathcal{B}^{(i)}$ ($i = 1, \dots, N + 1$) have the same combinatorial type exist, we first need to generalize Theorem 0.1.

Let \mathcal{Q} be a reduced quartic and choose a smooth point z_o on \mathcal{Q} such that

♣ the tangent line l_{z_o} is tangent to \mathcal{Q} at z_o with multiplicity 2 and meets \mathcal{Q} at two distinct residual points, or z_o is an inflection point of \mathcal{Q} and $I_{z_o}(l_{z_o}, \mathcal{Q}) = 3$.

Let \mathcal{C} be a contact conic to \mathcal{Q} . Then $f_{\mathcal{Q}}^*(\mathcal{C}) = \mathcal{C}^+ + \mathcal{C}^-$, $\nu_{z_o}^* \mathcal{C}^+$ gives rise to a section $s_{\mathcal{C}^+, z_o}$, or a bisection on $S_{\mathcal{Q}, z_o}$. By Theorem 2.1, we have a unique section for any horizontal divisor. We put

$$s_{z_o}(\mathcal{C}^+) := \begin{cases} s_{\mathcal{C}^+, z_o} & \text{if } z_o \in \mathcal{C}, \\ \tilde{\psi}(\nu_{z_o}^* \mathcal{C}^+) & \text{if } z_o \notin \mathcal{C}. \end{cases}$$

Here $\tilde{\psi}$ is the map defined in Theorem 2.1.

Theorem 0.3 *Let \mathcal{Q} be a 2-nodal quartic or a quartic with one tacnode. Then $f_{\mathcal{C}}^* \mathcal{Q}$ is splitting if and only if $s_{z_o}(\mathcal{C}^+)$ is 2-divisible in $\text{MW}(S_{\mathcal{Q}, z_o})$ for any z_o satisfying ♣.*

By using Theorem 0.3, we can prove:

Theorem 0.4 *There exist $\mathcal{B}^1, \dots, \mathcal{B}^{N+1}$ as in Theorem 0.2 with the same combinatorial type. Moreover, there exists a Zariski $N + 1$ -plet of degree $2N + 4$.*

In the case of $N = 2$, by combining with [3, Theorem 1.4], we have

Theorem 0.5 *There exists a Zariski 5-plet of degree 8 for a 2-nodal quartic and 2 smooth conics.*

Remark 0.2 In [6], the authors constructed a Zariski 4-plet of degree 10 consisting of a quartic and three conics. Theorem 0.5 gives an example of a Zariski 5-plet of degree 8. This apparently demonstrates that the geometry of contact conics is extremely complicated but rich and worth investigating.

The organization of this article is as follows:

In Section 1, we review our method to distinguish the topology of $(\mathbb{P}^2, \mathcal{B})$ considered in [4] and prove Theorem 0.2. The rest of this article is devoted to show the existence of conics $\mathcal{C}_j^{(i)}$ ($i = 1, \dots, N+1, j = 1, \dots, N$) such that $\mathcal{B}_1, \dots, \mathcal{B}_{N+1}$ form a Zariski $(N+1)$ -plet. In section 2, we introduce some sections of $S_{\mathcal{Q}, z_o}$ which are given by curves on \mathbb{P}^2 not passing through z_o . We prove Theorem 0.3 in Section 3. We review our method to deal with bisections given in [6] and give examples which proves Theorem 0.4 in Sections 4 and 5, respectively. Theorem 0.5 will be proved in section 5.

1 Proof of Theorem 0.2

Let \mathcal{Q} be an irreducible quartic and let $\mathcal{C}_1, \dots, \mathcal{C}_N$ be contact conics to \mathcal{Q} . Put $\underline{\mathcal{C}} = \sum_{i=1}^N \mathcal{C}_i$ and for $I(\neq \emptyset) \subset \{1, \dots, N\}$, put $\underline{\mathcal{C}}_I = \sum_{i \in I} \mathcal{C}_i$. Likewise [4] we define $\underline{\text{Sub}}(\mathcal{Q}, \underline{\mathcal{C}})$ as

$$\underline{\text{Sub}}(\mathcal{Q}, \underline{\mathcal{C}}) := \{\mathcal{Q} + \underline{\mathcal{C}}_I \mid I(\neq \emptyset) \subseteq \{1, \dots, N\}\},$$

and its subset $\underline{\text{Sub}}_k(\mathcal{Q}, \underline{\mathcal{C}})$ as

$$\underline{\text{Sub}}_k(\mathcal{Q}, \underline{\mathcal{C}}) := \{\mathcal{Q} + \underline{\mathcal{C}}_I \mid I(\neq \emptyset) \subseteq \{1, \dots, N\}, \#(I) = k\}.$$

Now we define the map

$$\Phi_{\mathcal{Q}, \underline{\mathcal{C}}}^1 : \underline{\text{Sub}}_1(\mathcal{Q}, \underline{\mathcal{C}}) \rightarrow \{0, 1\}$$

as follows:

$$\Phi_{\mathcal{Q}, \underline{\mathcal{C}}}^1(\mathcal{Q} + \mathcal{C}_j) = \begin{cases} 1 & \text{if } f_{\mathcal{C}_j}^* \mathcal{Q} \text{ is splitting} \\ 0 & \text{if } f_{\mathcal{C}_j}^* \mathcal{Q} \text{ is irreducible.} \end{cases}$$

Consider N more contact conics $\mathcal{C}'_1, \dots, \mathcal{C}'_N$ to \mathcal{Q} , and put $\underline{\mathcal{C}}' = \sum_{i=1}^N \mathcal{C}'_i$. If there exists a homeomorphism $h : (\mathbb{P}^2, \mathcal{Q} + \underline{\mathcal{C}}) \rightarrow (\mathbb{P}^2, \mathcal{Q} + \underline{\mathcal{C}}')$, then h induces a bijection $h_{\natural} : \underline{\text{Sub}}(\mathcal{Q}, \underline{\mathcal{C}}) \rightarrow \underline{\text{Sub}}(\mathcal{Q}, \underline{\mathcal{C}}')$. Moreover, h_{\natural} induces a bijection $\underline{\text{Sub}}_1(\mathcal{Q}, \underline{\mathcal{C}}) \rightarrow \underline{\text{Sub}}_1(\mathcal{Q}, \underline{\mathcal{C}}')$ such that $\Phi_{\mathcal{Q}, \underline{\mathcal{C}}}^1 = \Phi_{\mathcal{Q}, \underline{\mathcal{C}}'}^1 \circ h_{\natural}$:

$$\begin{array}{ccc} \underline{\text{Sub}}_1(\mathcal{Q}, \underline{\mathcal{C}}) & & \\ \downarrow h_{\natural} & \searrow \Phi_{\mathcal{Q}, \underline{\mathcal{C}}}^1 & \\ \underline{\text{Sub}}_1(\mathcal{Q}, \underline{\mathcal{C}}') & \xrightarrow{\Phi_{\mathcal{Q}, \underline{\mathcal{C}}'}^1} & \{0, 1\} \end{array}$$

Under these setting, we prove Theorem 0.2. Let $\mathcal{B}^i = \mathcal{Q} + \underline{\mathcal{C}}^{(i)}$, $\underline{\mathcal{C}}^{(i)} = \sum_{j=1}^N \mathcal{C}_j^{(i)}$, ($i = 1, \dots, N$) be the reduced curve as in Theorem 0.2. Then we have

$$\# \left((\Phi_{\mathcal{Q}, \underline{\mathcal{C}}^{(i)}}^1)^{-1}(1) \right) = N - i + 1.$$

Hence there exists no homeomorphism $h : (\mathbb{P}^2, \mathcal{B}^i) \rightarrow (\mathbb{P}^2, \mathcal{B}^k)$ for i, k ($i \neq k$), as \mathcal{Q} is preserved under any homeomorphism.

2 Elliptic surfaces

We here summarize some facts on elliptic surfaces. We refer to [9], [10] and [13] for details.

2.1 Some general settings and facts.

Throughout this article, an elliptic surface always means a smooth projective surface S with a fibration $\varphi : S \rightarrow C$ over a smooth projective curve, C , as follows:

- (i) There exists a *non empty* finite subset $\text{Sing}(\varphi) \subset C$ such that $\varphi^{-1}(v)$ is a smooth curve of genus 1 for $v \in C \setminus \text{Sing}(\varphi)$, while $\varphi^{-1}(v)$ is not a smooth curve of genus 1 for $v \in \text{Sing}(\varphi)$.
- (ii) There exists a section $O : C \rightarrow S$ (we identify O with its image in S).
- (iii) there is no exceptional curve of the first kind in any fiber.

For $v \in \text{Sing}(\varphi)$, we call $F_v = \varphi^{-1}(v)$ a singular fiber over v . As for the types of singular fibers, we use notation given by Kodaira ([9]). For $v \in \text{Sing}(\varphi)$, we denote the irreducible decomposition of F_v by

$$F_v = \Theta_{v,0} + \sum_{i=1}^{m_v-1} a_{v,i} \Theta_{v,i},$$

where m_v is the number of irreducible components of F_v and $\Theta_{v,0}$ denotes the irreducible component with $\Theta_{v,0}O = 1$. We call $\Theta_{v,0}$ *the identity component* of F_v . We also define a subset $\text{Red}(\varphi)$ of $\text{Sing}(\varphi)$ to be $\text{Red}(\varphi) := \{v \in \text{Sing}(\varphi) \mid F_v \text{ is reducible}\}$. For a section $s \in \text{MW}(S)$, s is said to be *integral* if $sO = 0$.

Let $\text{MW}(S)$ be the set of sections of $\varphi : S \rightarrow C$. By our assumption, $\text{MW}(S) \neq \emptyset$. On a smooth fiber F of φ , by regarding $F \cap O$ as the zero element, we can consider the abelian group structure on F . Hence for $s_1, s_2 \in \text{MW}(S)$, one can define the addition $s_1 \dot{+} s_2$ on $C \setminus \text{Sing}(\varphi)$. By [9, Theorem 9.1], $s_1 \dot{+} s_2$ can be extended over C , and we can consider $\text{MW}(S)$ as an abelian group. $\text{MW}(S)$ is called the Mordell-Weil group. We also denote the multiplication-by- m map ($m \in \mathbb{Z}$) on $\text{MW}(S)$ by $[m]s$ for $s \in \text{MW}(S)$. Note that $[2]s$ is the double of s *with respect to the group law on* $\text{MW}(S)$. On the other hand, we can regard the generic fiber $E := S_\eta$ of S as a curve of genus 1 over $\mathbb{C}(C)$, the rational function field of C . The restriction of O to E gives rise to a $\mathbb{C}(C)$ -rational point of E , and one can regard E as an elliptic curve over $\mathbb{C}(C)$, O being the zero element. By considering the restriction to the generic fiber for each sections, $\text{MW}(S)$ can be identified with the set of $\mathbb{C}(C)$ -rational points $E(\mathbb{C}(C))$. For $s \in \text{MW}(S)$, we denote the corresponding rational point by P_s . Conversely, for an element $P \in E(\mathbb{C}(C))$, we denote the corresponding section by s_P .

We also denote the addition and the multiplication-by- m map on $E(\mathbb{C}(C))$ by $P_1 \dot{+} P_2$ and $[m]P_1$ for $P_1, P_2 \in E(\mathbb{C}(C))$, respectively. Again, $[2]P$ is the double of P *with respect to the group law on* $E(\mathbb{C}(C))$.

Let $\text{NS}(S)$ be the Néron-Severi group of S and let T_φ denote the subgroup of $\text{NS}(S)$ generated by O , a fiber F and $\{\Theta_{v,1}, \dots, \Theta_{v,m_v-1} \mid (v \in \text{Red}(\varphi))\}$. By Shioda [13], we have

Theorem 2.1 ([13, Theorem 1.2, 1.3] *Under our assumptions,*

- (i) $\text{NS}(S)$ is torsion free, and
(ii) there exists a natural map $\tilde{\psi} : \text{Div}(S) \rightarrow \text{MW}(S)$ which induces an isomorphism of abelian groups

$$\psi : \text{NS}(S)/T_\varphi \cong \text{MW}(S).$$

A curve on S is said to be horizontal with respect to φ if it does not contain any fiber components of φ . For a horizontal curve D on S , we put $s(D) := \tilde{\psi}(D)$. A bisection of $\varphi : S \rightarrow C$ is a reduced horizontal curve D which intersects at two points with a general fiber of φ . As we see later, in order to construct reduced plane curve with prescribed property, we make use of a bisection D and a section s with $s(D) = s$, as we see later.

In [13], Shioda defined a \mathbb{Q} -valued bilinear form $\langle \cdot, \cdot \rangle$ on $\text{MW}(S)$ by using the intersection pairing on $\text{NS}(S)$ as follows:

- $\langle s, s \rangle \geq 0$ for $\forall s \in \text{MW}(S)$ and the equality holds if and only if s is an element of finite order in $\text{MW}(S)$.
- An explicit formula for $\langle s_1, s_2 \rangle$ ($s_1, s_2 \in \text{MW}(S)$) is given as follows:

$$\langle s_1, s_2 \rangle = \chi(\mathcal{O}_S) + s_1 O + s_2 O - s_1 s_2 - \sum_{v \in \text{Red}(\varphi)} \text{Contr}_v(s_1, s_2).$$

Here we identify sections with their images, i.e., curves on S , and the product denotes the intersection pairing on $\text{NS}(S)$. Also $\text{Contr}_v(s_1, s_2)$ is given by

$$\text{Contr}_v(s_1, s_2) = (s_1 \Theta_{v,1}, \dots, s_1 \Theta_{v,m_v-1})(-A_v)^{-1} \begin{pmatrix} s_2 \Theta_{v,1} \\ \vdots \\ s_2 \Theta_{v,m_v-1} \end{pmatrix},$$

where A_v is the intersection matrix $(\Theta_{v,i} \Theta_{v,j})_{1 \leq i,j \leq m_v-1}$.

As for explicit values of $\text{Contr}_v(s_1, s_2)$, we refer to [13, (8.16)]. As for explicit structures of $\text{MW}(S)$ for rational elliptic surfaces, see [12].

2.2 Rational elliptic surfaces, $S_{\mathcal{Q}, z_o}$

A surface S is called a rational elliptic surface if it is an elliptic surface birationally equivalent to \mathbb{P}^2 .

It is well-known that any rational elliptic surface is realized as a double cover of the Hirzebruch surface of degree 2, which we denote by Σ_2 (see, for example, [6, 2.2.2], [10, Lecture 3, 2], [20, 1.2]). The branch curve is of the form $\Delta_0 + \mathcal{T}$, where Δ_0 is the section with $\Delta_0^2 = -2$ and $\mathcal{T} \sim 3(\Delta_0 + \mathfrak{f})$, \mathfrak{f} being a fiber of the ruling of Σ_2 . Note that the preimage of Δ_0 is a section O .

On the other hand, for a reduced quartic \mathcal{Q} , which is not a union of four concurrent and a general point z_o of \mathcal{Q} , we can associate a rational elliptic surface $S_{\mathcal{Q}, z_o}$ as in the Introduction.

As we have seen in [20, Section 4], if z_o satisfies \clubsuit , the tangent line l_{z_o} at z_o becomes an irreducible component of a singular fiber of type either I_2 or III . For other singular fibers, see [11, Section 6].

Note that $\widehat{\mathbb{P}^2}$ can be blown down to Σ_2 and we denote the composition of blowing-downs by $\bar{q} : \widehat{\mathbb{P}^2} \rightarrow \Sigma_2$. This is nothing but the realization of $S_{\mathcal{Q}, z_o}$ as a double cover of Σ_2 .

2.3 Sections, bisections of $S_{\mathcal{Q}, z_o}$ and contact conics to \mathcal{Q}

We here recall our method to treat bisections of $S_{\mathcal{Q}, z_o}$ considered in [6, 2.2.3]. Choose homogeneous coordinates $[T : X : Z]$ of \mathbb{P}^2 such that $z_o = [0 : 1 : 0]$, and the tangent line at z_o is given by $Z = 0$. \mathcal{Q} is given by a homogeneous polynomial of the form

$$\mathcal{Q} : F(T, X, Z) = X^3 Z + b_2(T, Z)X^2 + b_3(T, Z)X + b_4(T, Z) = 0,$$

where $b_i(T, Z)$ ($i = 2, 3, 4$) are homogeneous polynomials of degree i . Let U be an affine open set of \mathbb{P}^2 with coordinate $(t, x) = (T/Z, X/Z)$. Under these circumstances, the elliptic curve $E_{\mathcal{Q}, z_o}$ over $\mathbb{C}(t) (\cong \mathbb{C}(\mathbb{P}^1))$ is given by the Weierstrass equation:

$$y^2 = F(t, x, 1).$$

Choose an element $P = (x(t), y(t)) \in E_{\mathcal{Q}, z_o}(\mathbb{C}(t))$ and $r(t) \in \mathbb{C}(t)$. Consider the line L in $\mathbb{A}_{\mathbb{C}(t)}^2$ defined by

$$L : y = l(t, x) = r(t)(x - x(t)) + y(t).$$

Then we have

$$F(t, x, 1) - \{l(t, x)\}^2 = (x - x(t))g(t, x),$$

where $g(t, x) \in \mathbb{C}(t)[x]$ with $\deg_x g(t, x) = 2$. Here, $g(t, x)$ can be considered as a rational function on Σ_2 . The zero divisor $(g(t, x))_0$ of $g(t, x)$ is of the form

$$C_{g(t, x)} + (\text{fiber components}),$$

where $C_{g(t, x)}$ is a curve without any fiber components, $C_{g(t, x)} \cdot \mathbf{f} = 2$ and $\Delta_0 \not\subset C_{g(t, x)}$.

Lemma 2.1 ([6, Lemma 2]) *If $C_{g(t, x)}$ is irreducible and $C_{g(t, x)} \not\subset \mathcal{T}$, then*

(i) *$(q \circ f)^* C_{g(t, x)}$ is of the form*

$$(q \circ f)^* C_{g(t, x)} = C_{g(t, x)}^+ + C_{g(t, x)}^- + E,$$

where $C_{g(t, x)}^\pm$ are bisections and E is a divisor whose irreducible components are supported on the exceptional set of μ .

(ii) *If we choose $C_{g(t, x)}^+$ suitably,*

$$(y - l(t, x))|_{E_{\mathcal{Q}, z_o}} = C_{g(t, x)}^+|_{E_{\mathcal{Q}, z_o}} + P - 3O,$$

which implies $P_{C_{g(t, x)}^+} = -P$ or equivalently $s(C_{g(t, x)}^+) = -s_P$.

Definition 2.1 For a given $P = (x(t), y(t)) \in E_{\mathcal{Q}, z_o}(\mathbb{C}(t))$ and $r(t) \in \mathbb{C}(t)$, we denote the irreducible bisection $C_{g(t, x)}^+$ obtained in Lemma 2.1 by $D(r(t), P)$. We also denote a plane curve $\bar{q} \circ f(D(r(t), P))$ by $C(r(t), P)$.

In [6], we applied Lemma 2.1 to the case when $x(t), y(t) \in \mathbb{C}[t]$ and a suitable $r(t) \in \mathbb{C}[t]$ in order to find contact conics to \mathcal{Q} .

In this article, we take the same approach to find explicit example and to prove Theorem 0.4 as that in [6].

3 Geometrically useful sections of $S_{\mathcal{Q}, z_o}$

3.1 Distinguished point free section

Let \mathcal{Q} be a reduced quartic which has at least one non-linear irreducible component. Choose a general point z_o satisfying \clubsuit . Let $\varphi_{\mathcal{Q}, z_o} : S_{\mathcal{Q}, z_o} \rightarrow \mathbb{P}^1$ be the rational elliptic surface associated to \mathcal{Q} and z_o . By our choice of z_o , $S_{\mathcal{Q}, z_o}$ has a singular fiber F_∞ of type I_2 or III , which we denote by

$$F_\infty = \Theta_{\infty, 0} + \Theta_{\infty, 1},$$

where $\Theta_{\infty, 0}$ is the exceptional divisor of the first blowing-up of $\nu : S_{\mathcal{Q}, z_o} \rightarrow S_{\mathcal{Q}}$ and $\Theta_{\infty, 1}$ arises from the tangent line at z_o , l_{z_o} .

Let us start with the following lemma:

Lemma 3.1 *Let $s \in \text{MW}(S_{\mathcal{Q}, z_o})$ such that $sO = 0$ and $s\Theta_{\infty, 1} = 1$. Then $\bar{q} \circ f(s)$ is a line L_s in \mathbb{P}^2 such that*

- (i) $z_o \notin L_s$, and
- (ii) $I_x(L_s, \mathcal{Q})$ is even for $\forall x \in L_s \cap \mathcal{Q}$.

Conversely, any line satisfying the above two conditions gives rise to sections s_{L^\pm} such that $s_{L^\pm}O = 0$ and $s_{L^\pm}\Theta_{\infty, 1} = 1$.

Proof. By our construction, $\bar{q} \circ f(s)$ intersects any line through z_o at one point. Hence $\bar{q} \circ f(s)$ is a line not passing through z_o . Also, if $I_x(L_s, \mathcal{Q})$ is odd for some $x \in L_s \cap \mathcal{Q}$, $(\bar{q} \circ f)^*(L_s)$ is a sum of an irreducible curve which is mapped to L_s and exceptional curves for μ . On the other hand, both s and $[-1]_{\varphi_{\mathcal{Q}, z_o}} s$ are contained in $(\bar{q} \circ f)^* L_s$, which leads us to a contradiction.

Conversely, if L is a line satisfying the conditions (i) and (ii), then $(f' \circ \mu)^* L$ is of the form

$$(\bar{q} \circ f)^* L = L^+ + L^- + E,$$

where E consist of exceptional curves for $\mu : S_{\mathcal{Q}} \rightarrow S'_{\mathcal{Q}}$. Since L^\pm meet $(f' \circ \mu)^* l$ (l is a line through z_o) at one point. Hence $\nu^* L^\pm$ give rise to sections, which we denote by s_{L^\pm} . As $z_o \notin L$, $s_{L^\pm}O = 0$ and $s_{L^\pm}\Theta_{\infty, 1} = 1$. \square

Definition 3.1 For $s \in \text{MW}(S_{\mathcal{Q}, z_o})$, we call s a *distinguished point free section (dp-free section, for short)* if $sO = 0$ and $s\Theta_{\infty, 1} = 1$. Otherwise, we call s a *distinguished point sensitive section (dp-sensitive section, for short)*.

Remark 3.1 By Lemma 3.1, any distinguished point free section of $S_{\mathcal{Q}, z_o}$ arises from a line in \mathbb{P}^2 satisfying the conditions (i) and (ii).

3.2 Examples

Example 3.1 ([15]) For a smooth quartic curve \mathcal{Q} , it is well-known that \mathcal{Q} has 28 bitangent lines, which give rise to 56 dp-free sections, which generate $\text{MW}(S_{\mathcal{Q}, z_o})$. In [15], these 56 sections are intensively studied and their applications are given. See [15], for detail.

Example 3.2 Let \mathcal{Q} be an irreducible quartic with 2 nodes only as its singularities. Choose a smooth point z_o satisfying the condition (\clubsuit). In this case, The configuration of reducible singular fibers of $\varphi_{\mathcal{Q},z_o} : S_{\mathcal{Q},z_o} \rightarrow \mathbb{P}^1$ is either $3I_2$ or $2I_2, III$. For both cases, $\text{MW}(S_{\mathcal{Q},z_o}) \cong A_1^* \oplus D_4^*$. We consider a basis of $\text{MW}(S_{\mathcal{Q},z_o})$ consisting of dp-free sections.

Let L_0 be the line connecting x_1 and x_2 . L_0 gives rise to 2 dp-free sections s_0^+ and s_0^- .

By considering the projection centered at x_i ($i = 1, 2$), there exist 4 tangent lines $L_{i,j}$ ($j = 1, 2, 3, 4$) through x_i . Here, the case when $L_{i,j}$ is tangent to one of the branches of the node with multiplicity 3 is also considered. Since

$$(q \circ f_{\mathcal{Q}})^* L_{i,j} = L_{i,j}^+ + L_{i,j}^-,$$

these 8 lines give rise to 16 dp-free sections $s_{i,j}^+ = \nu_{z_o}^* L_{i,j}^+$, $s_{i,j}^- = \nu_{z_o}^* L_{i,j}^-$ ($i = 1, 2, j = 1, 2, 3, 4$). By (re)labeling i, j, \pm suitably, we may assume that

$$s_{1,k}^+ s_{1,l}^+ = -\delta_{kl}, \quad s_{1,j}^+ s_{2,1}^+ = 1 \quad (j = 1, 2, 3, 4), \quad s_0^\pm s_{i,j}^\pm = 0 \quad (i = 1, 2, j = 1, 2, 3, 4)$$

Hence, we have

$$\langle s_{1,k}^+, s_{1,l}^+ \rangle = \delta_{kl}, \quad \langle s_{1,k}^+, s_{2,1}^+ \rangle = -\frac{1}{2}, \quad \langle s_0^+, s_0^+ \rangle = \frac{1}{2}, \quad \langle s_0^+, s_{i,j}^+ \rangle = 0 \quad (i = 1, 2, j = 1, 2, 3, 4)$$

Put

$$s_{z_o,0} = s_0^+ = \nu_{z_o}^* L_0^+, \quad s_{z_o,i} = s_{1,i}^+ = \nu_{z_o}^* L_{1,i}^+, \quad (i = 1, 2, 3) \quad s_{z_o,4} = s_{2,1}^+ = \nu_{z_o}^* L_{2,1}^+.$$

Then we have the Gram matrix $[\langle s_{z_o,i}, s_{z_o,j} \rangle]$

$$\begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & 1 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & 1 & -\frac{1}{2} \\ 0 & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 1 \end{bmatrix}.$$

Let M be a sublattice of $\text{MW}(S_{\mathcal{Q},z_o})$ given by $s_{z_o,0}, s_{z_o,1}, s_{z_o,2}, s_{z_o,3}, s_{z_o,4}$. As $\det M = 1/8$. $\text{MW}(S_{\mathcal{Q},z_o}) = M$. Finally if we put $t_0 = s_{z_o,0}$, $t_1 = s_{z_o,1}$, $t_2 = s_{z_o,1} + s_{z_o,2}$, $t_3 = -(s_{z_o,3} + s_{z_o,4})$, $t_4 = -s_{z_o,4}$, we get a basis for $A_1^* \oplus D_4^*$ with the well known Gram matrix.

Example 3.3 Let \mathcal{Q} be an irreducible quartic with one tacnode, x_o . Choose $z_o \in \mathcal{Q}$ satisfying \clubsuit and let $\varphi_{\mathcal{Q},z_o} : S_{\mathcal{Q},z_o} \rightarrow \mathbb{P}^1$ be the rational elliptic surface as in the Introduction. The configuration of reducible singular fibers of $\varphi_{\mathcal{Q},z_o}$ is either I_4, I_2 or I_4, III . By [12], $\text{MW}(S_{\mathcal{Q},z_o}) \cong A_1^* \oplus A_3^*$. We also consider a basis consisting of do-free sections as in Example 3.2.

Let L_{x_o} be the tangent line at x_o . By a similar argument to the one in Example 3.2, there exist 4 lines L_i ($i = 1, \dots, 4$) through x_o and tangent to \mathcal{Q} at another residual point. By our construction of $S_{\mathcal{Q}}$, we have

$$(q \circ f_{\mathcal{Q}})^*(L_{x_o}) = L_{x_o}^+ + L_{x_o}^- + E_{x_o}, \quad (q \circ f_{\mathcal{Q}})^*(L_i) = L_i^+ + L_i^- + E_i, \quad (i = 1, \dots, 4),$$

where E_{x_o}, E_i ($i = 1, \dots, 4$) are divisors whose supports are contained in the exceptional set of μ . By Lemma 3.1, we infer that $\nu_{z_o}^* L_{x_o}^\pm, \nu_{z_o}^* L_i^\pm$ ($i = 1, \dots, 4$) are dp-free sections of $\varphi_{\mathcal{Q}, z_o}$. Put

$$s_{z_o,0} := \nu_{z_o}^* L_{x_o}^+, \quad s_{z_o,i} := \nu_{z_o}^* L_i^+.$$

By labelling appropriately, we may assume:

$$\begin{aligned} \langle s_{z_o,0}, s_{z_o,0} \rangle &= \frac{1}{2}, & \langle s_{z_o,0}, s_{z_o,i} \rangle &= 0, \quad (i = 1, \dots, 4) \\ \langle s_{z_o,i}, s_{z_o,i} \rangle &= \frac{3}{4}, & \langle s_{z_o,i}, s_{z_o,j} \rangle &= -\frac{1}{4}, \quad (i \neq j) \quad (i, j = 1, \dots, 4) \end{aligned}$$

Hence the Gram matrix for $s_{z_o,i}$ ($i = 0, \dots, 3$) is

$$\begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{3}{4} & -\frac{1}{4} & -\frac{1}{4} \\ 0 & -\frac{1}{4} & \frac{3}{4} & -\frac{1}{4} \\ 0 & -\frac{1}{4} & -\frac{1}{4} & \frac{3}{4} \end{bmatrix}$$

Since the determinant of the above matrix is $1/8$, we infer that $s_{z_o,i}$ ($i = 0, \dots, 3$) form a basis of $\text{MW}(S_{\mathcal{Q}, z_o})$. Now put $t_0 := s_{z_o,0}, t_1 := s_{z_o,1}, t_2 := s_{z_o,1} + s_{z_o,2}, t_3 := -s_{z_o,3}$. The Gram matrix $[\langle t_i, t_j \rangle]$ becomes the well known one.

Example 3.4 Let \mathcal{Q} be an irreducible quartic with 3 nodes, which we denote x_1, x_2, x_3 . Choose z_o satisfying \clubsuit in the Introduction. In this case, the configuration of singular fibers of $\varphi_{\mathcal{Q}, z_o} : S_{\mathcal{Q}, z_o} \rightarrow \mathbb{P}^1$ is either 4I_2 or $3\text{I}_2, \text{III}$. For both cases, $\text{MW}(S_{\mathcal{Q}, z_o}) \cong (A_1^*)^{\oplus 4}$. As we have seen in [21], a line connecting 2 nodes x_i and x_j gives rise to two dp-free sections $s_{i,j}^\pm$, which are generators of one of the direct summands of $\text{MW}(S_{\mathcal{Q}, z_o})$. Hence 3 direct summands are generated these dp-free sections which arise from lines connecting 2 nodes. We then choose a line l_4 which passes through x_3 and tangent to \mathcal{Q} another point or passes through x_3 with multiplicity 4. Then l_4 gives another dp-free section s_4^\pm . By relabelling $s_{i,j}^\pm$ suitably, we may assume that

$$s_4^+ s_{i,j}^+ = 0.$$

Put $s_1 := s_{1,2}^+, s_2 := s_{1,3}^+, s_3 := s_{2,3}^+, s_4 := s_4^+$, and we have the Gram matrix

$$[\langle s_i, s_j \rangle] = \begin{bmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 & 1 \end{bmatrix}$$

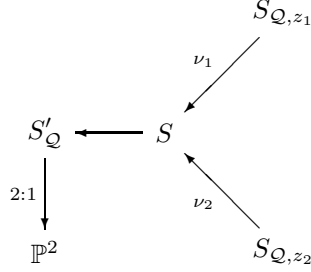
Since the determinant of the above matrix is $1/16$, we infer that s_1, \dots, s_4 form a basis of $\text{MW}(S_{\mathcal{Q}, z_o})$.

Remark 3.2 *It is an interesting problem to determine which rational elliptic surfaces admit a dp-free basis, and also to find an explicit geometric description of each basis.*

4 Proof of Theorem 0.3

4.1 An application of distinguished point free sections

We first consider an application of distinguished point free sections. Let \mathcal{Q} be as before. Let $f'_Q : S'_Q \rightarrow \mathbb{P}^2$ be the double cover branched along \mathcal{Q} and S_Q the canonical resolution of the singularities of S'_Q . Choose two distinguished point z_1 and z_2 satisfying \clubsuit . The resolution maps for the pencils Λ_{z_1} and Λ_{z_2} will be denoted by $\nu_i : S_{Q,z_i} \rightarrow S_Q$, respectively. Each ν_i is a composition of two blowing-ups.



The exceptional divisor of the second blowing-up of ν_1 (resp. ν_2) gives rise to a section of S_{Q,z_1} (resp. S_{Q,z_2}). We will regard this section as the zero section and denote it by O_1 (resp. O_2). By construction, S_{Q,z_1} and S_{Q,z_2} are rational elliptic surfaces that have the same configuration of singular fibers, except possibly the one arising from tangent lines l_{z_i} at z_i which is either of type I_2 or III .

Let D_1, \dots, D_m be divisors on S such that they do not pass through $(f'_Q \circ \nu_1)^{-1}(z_1), (f'_Q \circ \nu_2)^{-1}(z_2)$ and their strict transforms under ν_1 (resp. ν_2) give rise to sections of S_{Q,z_1} (resp. S_{Q,z_2}). Note that these sections are distinguished point free sections. Let $s_i(D_j)$ denote the section corresponding to D_j on S_{Q,z_i} . Put $C_i = s_i(D_1) + \dots + s_i(D_m) \in \text{MW}(S_{Q,z_i})$. Put $\overline{C_i} = \nu_i(C_i)$ and let $\widehat{C_i}$ be the strict transform of $\overline{C_i}$ under ν_j^{-1} . ($i \neq j$). For z_1 and z_2 with \clubsuit , $\widehat{C_2}$ becomes a multi-section of S_{z_1} . Under this setting, we have:

Theorem 4.1 *Suppose that $C_2 \neq O_2$ and $z_1 \notin \overline{C_2}$. Then*

$$s(\widehat{C_2}) = C_1.$$

Proof. Note that each surface S_{Q,z_i} has a I_2 or III type singular fiber whose components arise from the exceptional divisor of the first blow up in ν_i which meets O_i and the strict transform of the tangent line l_{z_i} of \mathcal{Q} at z_i . We will denote these components by $\Theta_{z_i,0}$ and $\Theta_{z_i,1}$. All the other reducible singular fibers arise from the exceptional sets of the resolution $S_Q \rightarrow S'_Q$, hence they are in 1 to 1 correspondence with the singularities of \mathcal{Q} . We will denote their components by $\Theta_{v,i}$ where $v \in \text{Sing}(\mathcal{Q})$. Let $D' = s_1(D_1) + \dots + s_1(D_m)$. Note that the sum taken here is regarded as a sum of divisors on S_{Q,z_1} . Then since the Abel-Jacobi map $\tilde{\psi}_i$ for each surface is a homomorphism, $\tilde{\psi}_1(D') = s_1(D_1) + \dots + s_1(D_m) = C_1$. Hence by [13, Lemma 5.1] we have the equivalence

$$D' \underset{S_{Q,z_1}}{\sim} (C_1) + d(O_1) + nF_1 + a\Theta_{z_1,1} + \sum_{v \in \text{Sing}(\mathcal{Q})} \sum_{i=1}^{m_v-1} b_{v,i}\Theta_{v,i},$$

where $\sim_{S_{\mathcal{Q}, z_1}}$ denotes linear equivalence of divisors on $S_{\mathcal{Q}, z_1}$

Similarly for $D'' = s_2(D_1) + \cdots + s_2(D_m)$, we have the equivalence

$$D'' \sim_{S_{\mathcal{Q}, z_2}} (C_2) + d(O_2) + nF_2 + a\Theta_{z_2, 1} + \sum_{v \in \text{Sing}(\mathcal{Q})} \sum_{i=1}^{m_v-1} b_{v,i} \Theta_{v,i}.$$

Note that by construction of D', D'' and [9, Theorem 9.1] (or [19, 1.1], the coefficients $d, n, a, b_{v,i}$ are the same in both cases. Since $\nu_{1*}(D') = D_1 + \cdots + D_m = \nu_{2*}(D'')$, by [7, Theorem 1.4], we have

$$\begin{aligned} \bar{C}_1 + nF_1 + a\nu_*\Theta_{z_1, 1} + \sum_{v \in \text{Sing}(\mathcal{Q})} \sum_{i=1}^{m_v-1} b_{v,i} \Theta_{v,i} \\ \sim_{S_{\mathcal{Q}}} \bar{C}_2 + nF_2 + a\nu_*\Theta_{z_2, 1} + \sum_{v \in \text{Sing}(\mathcal{Q})} \sum_{i=1}^{m_v-1} b_{v,i} \Theta_{v,i}. \end{aligned}$$

Then since $F_1 \sim_{S_{\mathcal{Q}}} F_2$ and $\Theta_{z_1, 1} \sim_{S_{\mathcal{Q}}} \Theta_{z_2, 1}$, because they are inverse images of lines of \mathbb{P}^2 , we obtain the equivalence

$$\bar{C}_1 \sim_{S_{\mathcal{Q}}} \bar{C}_2.$$

By pulling this equivalence back by ν_1 , we obtain

$$\begin{aligned} \hat{C}_2 \sim_{S_{\mathcal{Q}, z_1}} C_1 + \alpha O_1 + \beta \Theta_{z_1, 0} \\ \sim_{S_{\mathcal{Q}, z_1}} C_1 + \alpha O_1 + \beta F - \beta \Theta_{z_1, 1}, \end{aligned}$$

for some integers α and β . Hence by Theorem 2.1 we have $\tilde{\psi}_1(\hat{C}_2) = C_1$. \square

4.2 Proof of Theorem 0.3

Let \mathcal{Q} be a quartic and $z_o \in \mathcal{Q}$ be a point satisfying \clubsuit . Furthermore, assume that $\text{MW}(S_{\mathcal{Q}, z_o})$ is generated by dp-free sections $s_{z_o, i}$ ($i = 0, \dots, k$) as in the setting of the previous subsection. Let \mathcal{C}' be a contact conic with $z_o \notin \mathcal{C}'$. On $S_{\mathcal{Q}}$, $(q \circ f_{\mathcal{Q}})^*\mathcal{C}' = \mathcal{C}'^+ + \mathcal{C}'^-$, and $\nu_{z_o}^*\mathcal{C}'^{\pm}$ become bisections on $S_{\mathcal{Q}, z_o}$. Let $s_{z_o}(\nu_{z_o}^*\mathcal{C}'^+) (= \tilde{\psi}(\nu_{z_o}^*\mathcal{C}'^+))$ be the section determined by \mathcal{C}'^+ . Assume that

$$s_{z_o}(\nu_{z_o}^*\mathcal{C}'^+) = \sum_{i=0}^k [a_i] s_{z_o, i}, \quad a_i \in \mathbb{Z}.$$

Choose $z'_o \in \mathcal{C}' \cap \mathcal{Q}$. Then $\nu_{z'_o}^*\mathcal{C}'^+$ is an element of $\text{MW}(S_{\mathcal{Q}, z'_o})$, which we denote by $s_{z'_o, \mathcal{C}'^+}$. Then we have

Proposition 4.1 *Under the above setting,*

$$s_{z'_o, \mathcal{C}'^+} = \sum_{i=0}^k [a_i] s_{z'_o, i}.$$

Proof. Put $s_{z'_o, C'^+} = \sum_{i=0}^4 [b_i] s_{z'_o, i}$. By Theorem 4.1, we have

$$s_{z_o}(\nu_{z_o}^* C'^+) = \sum_{i=0}^4 [b_i] s_{z_o, i}, \quad b_i \in \mathbb{Z}.$$

By our assumption, we have $a_i = b_i$ ($i = 0, \dots, k$). □

Case I: \mathcal{Q} is a 2-nodal quartic.

By Example 3.2, $\text{MW}(\mathcal{Q}_{z_o})$ is generated by bp-free sections. Furthermore, since $\langle s_{z'_o, C'^+}, s_{z'_o, C'^+} \rangle = 2$ and $\text{MW}(S_{\mathcal{Q}, z'_o}) \cong A_1^* \oplus D_4^*$, we have

Corollary 4.1 $f_{\mathcal{C}}^* \mathcal{Q}$ is split if and only if $a_0 = 2, a_1 = \dots a_4 = 0$.

Also by [18, Theorem 0.2], we have

Corollary 4.2 *There exists a Galois cover of \mathbb{P}^2 such that the Galois group is isomorphic to the dihedral group of order $2p$ and the branch locus is $\mathcal{C} + \mathcal{Q}$ if and only if $a_0 = 2, a_1 = \dots = a_4 = 0$.*

Case II: \mathcal{Q} is an irreducible quartic with a tacnode.

By Example 3.3, $\text{MW}(\mathcal{Q}_{z_o})$ is generated by bp-free sections. Furthermore, since $\langle s_{z'_o, C'^+}, s_{z'_o, C'^+} \rangle = 2$ and $\text{MW}(S_{\mathcal{Q}, z'_o}) \cong A_1^* \oplus A_3^*$, we have

Corollary 4.3 $f_{\mathcal{C}}^* \mathcal{Q}$ is split if and only if $a_0 = 2, a_1 = a_2 = a_3 = 0$.

Also by [18, Theorem 0.2], we have

Corollary 4.4 *There exists a Galois cover of \mathbb{P}^2 such that the Galois group is isomorphic to the dihedral group of order $2p$ and the branch locus is $\mathcal{C} + \mathcal{Q}$ if and only if $a_0 = 2, a_1 = a_2 = a_3 = 0$.*

The above Corollaries combined proves Theorem 0.3.

5 Proof of Theorem 0.4: Existence of contact conics

In this section we prove Theorem 0.4 by explicitly constructing the desired contact conics. First, we describe the method of constructing contact conics in general, and afterwards give explicit equations.

We utilise the method to construct bisections described in Section 2.3. We assume that $S_{\mathcal{Q}, z_o}$ is given the Weierstrass equation as in section 2.3.

Case I: \mathcal{Q} is a 2-nodal quartic.

Let P_i ($i = 0, \dots, 4$) be points in $E_{\mathcal{Q}, z_o}(\mathbb{C}(t))$ corresponding to the sections $s_{z_o, i}$ ($i = 0, \dots, 4$) in Example 3.2. Put $[2]P_0 = (x_1(t), y_1(t))$, $P_1 + P_2 = (x_2(t), y_2(t))$. We apply the method to construct bisections described in section 2.3 as follows:

- We choose $r_{i,a}(t) \in \mathbb{C}(t), a \in \mathbb{C}$ ($i = 1, 2$) appropriately such that both $C(r_{a,1}, [2]P_0)$ and $C(r_{a,2}, P_1 + P_2)$ give conics.

- Choose $a_1, \dots, a_N, b_1, \dots, b_N \in \mathbb{C}$ such that $C(r_{a_i,1}, [2]P_0)$ and $C(r_{b_i,2}, P_1 \dot{+} P_2)$ are contact conics to \mathcal{Q} contact at 4 distinct points.
- The $2N$ conics as above meets transversely with each other.

Put $\mathcal{C}_i = C(r_{a_i,1}, [2]P_0)$ ($i = 1, \dots, N$) and $\mathcal{C}'_j := C(r_{b_j,2}, P_1 \dot{+} P_2)$ ($j = 1, \dots, N$). Then we have

Lemma 5.1 *Under the setting given above,*

- (i) $f_{\mathcal{C}_i}^* \mathcal{Q}$ is splitting.
- (ii) $f_{\mathcal{C}'_j}^* \mathcal{Q}$ is irreducible.

Proof. (i) Choose $z'_o \in \mathcal{Q} \cap \mathcal{C}_i$. Note that \mathcal{C}_i is of the form $\mathcal{C}^+ + \mathcal{C}^-$ on $S_{\mathcal{Q}}$. By Proposition 4.1, we may assume that $\nu_{z'_o}^* \mathcal{C}^+ = [2]s_{z'_o,0}$. Hence by Theorem 0.1, $f_{\mathcal{C}_i}^* \mathcal{Q}$ is splitting.

(ii) Choose $z'_o \in \mathcal{Q} \cap \mathcal{C}'_j$. Again, \mathcal{C}'_j is of the form $\mathcal{C}^+ + \mathcal{C}^-$ on $S_{\mathcal{Q}}$. By Proposition 4.1, we may assume that $\nu_{z'_o}^* \mathcal{C}^+ = s_{z'_o,1} \dot{+} s_{z'_o,2}$. Hence by Theorem 0.1, $f_{\mathcal{C}'_j}^* \mathcal{Q}$ is irreducible. \square

Now in order to show the existence of conics as in Theorem 0.4, we choose N -conics from the above $2N$ conics suitably.

Case II: \mathcal{Q} is an irreducible quartic with one tacnode.

Let P_i ($i = 0, \dots, 3$) be points in $E_{\mathcal{Q}, z_o}(\mathbb{C}(t))$ corresponding to the sections $s_{z_o, i}$ ($i = 0, \dots, 4$) in Example 3.3. Put $[2]P_0 = (x_1(t), y_1(t))$, $P_1 \dot{+} P_2 = (x_2(t), y_2(t))$. We apply the method to construct bisections described in section 2.3. The remaining argument is almost the same as Case I, and we omit it.

5.1 Case I: \mathcal{Q} is a 2-nodal quartic

Let $F(T, X, Z)$ be a homogeneous polynomial

$$X^3 Z + (271350Z - 98T)X^2 + T(T - 5825Z)(T - 2025Z)X + 36T^2(T - 2025Z)^2$$

and let \mathcal{Q} be a quartic given by $F = 0$. \mathcal{Q} is a 2-nodal quartic and it has two nodes at $x_1 = [0 : 0 : 1]$ and $x_2 = [2025 : 0 : 1]$. Let $z_o = [0 : 1 : 0]$. The associated rational elliptic surface $S_{\mathcal{Q}, z_o}$ was given and studied by Shioda and Usui in [16, p. 198]. According to [16], $S_{\mathcal{Q}, z_o}$ has two singular fibers of type I_2 and one singular fiber of type III. The Mordell-Weil lattice of S is $\text{MW}(S) \cong A_1^* \oplus D_4^*$ and the narrow Mordell-Weil lattice is $\text{MW}(S)^0 \cong A_1 \oplus D_4$.

Let $L_0, L_{1,1}, L_{1,2}, L_{1,3}, L_{2,1}$ be lines defined by

$$\begin{aligned} L_0 : X &= 0, & L_{1,1} : 32T + X &= 0, & L_{1,2} : 28T - X &= 0, \\ L_{1,3} : 20T + X &= 0, & L_{2,1} : 35T + X - 70875Z &= 0. \end{aligned}$$

The lines $L_{i,j}$ pass through x_i and is tangent to \mathcal{Q} .

The lines give rise to dp-free sections s_0, \dots, s_4 with coordinates

$$\begin{aligned} s_0 &= (0, 6t(t - 2025)), & s_1 &= (-32t, 2t(t - 3465)), & s_2 &= (28t, 8t(t - 3285)), \\ s_3 &= (-20t, 4t(t - 1125)), & s_4 &= (-35t + 70875, (t + 20475)(t - 2025)), \end{aligned}$$

which are nothing but the basis given in [16]. By the explicit formula for the hight pairing, the Gram matrix with respect to this basis is the first matrix in Example 3.2. We now apply our observation in Section 2.3.

Consider

$$\begin{aligned} P &= [2]P_0 = (x_1(t), y_1(t)) \\ &= \left(\frac{1}{144}t^2 + \frac{1231}{72}t - \frac{5143775}{144}, -\frac{1}{1728}t^3 - \frac{2335}{576}t^2 + \frac{13493375}{576}t - \frac{29962489375}{1728} \right) \end{aligned}$$

and $r_{a,1}(t) = -t/12 + a$. By applying the method given in section 2.3, then we have

$$\begin{aligned} &F(t, x, 1) - \{l_1(t, x)\}^2 \\ &= -\frac{1}{2985984} (t^2 + 2462t - 144x - 5143775) \\ &\quad (144a^2t^2 + 354528a^2t - 20736xa^2 + 109032at^2 + 3456xta - 740703600a^2 \\ &\quad - 848072400ta - 86856575t^2 - 1677600tx + 20736x^2 + 719099745000a \\ &\quad + 328131278750t + 4886010000x - 174531500609375), \end{aligned}$$

where $l_1(t, x) = r_{a,1}(t)(x - x_1(t)) + y_1(t)$. Hence $C_{a,1} = C(r_{a,1}(t), [2]P_0)$ is a plane curve of degree 2 given as the zero locus of the second factor.

Next we take

$$\begin{aligned} P &= P_1 + P_2 = (x_2(t), y_2(t)) \\ &= \left(\frac{1}{36}t^2 + \frac{435}{2}t - \frac{921375}{4}, -\frac{1}{216}t^3 - \frac{41625}{8}t + \frac{373156875}{8} \right) \end{aligned}$$

and $r_{b,2}(t) = -t/6 + b$. By applying the method in section 2.3 again, we have

$$\begin{aligned} &F(t, x, 1)^2 - l_2(t, x)^2 \\ &= -\frac{1}{5184} (t^2 + 7830t - 36x - 8292375) \\ &\quad (4b^2t^2 + 31320b^2t - 144b^2x + 3732bt^2 + 48btx - 33169500b^2 + 12555000bt + 683865t^2 \\ &\quad + 17208tx + 144x^2 - 13433647500b + 1258071750t + 5904900x - 1360156809375), \end{aligned}$$

where $l_2(t, x) = r_{b,2}(t)(x - x_2(t)) + y_2(t)$. Hence $C_{b,2} = C(r_{b,2}(t), P_1 + P_2)$ is a plane curve of degree 2 given as the zero locus of the second factor.

By straightforward computation with computer similar to that in [6, Lemma 5], one can check that the three condition in the beginning of this section are satisfied. Therefore, we have a Zariski $N + 1$ -plet.

5.2 Case II: \mathcal{Q} is an irreducible quartic with one tacnode

Let $F(T, X, Z)$ be a homogeneous polynomial

$$X^3Z + (25T + 9Z)X^2Z + (144T^2Z + T^3)X + 16T^4$$

and let \mathcal{Q} be a quartic given by $F = 0$. \mathcal{Q} is an irreducible quartic with one tacnode at $x_o = [0 : 0 : 1]$. Let $z_o = [0 : 1 : 0]$. The associated rational elliptic surface $S_{\mathcal{Q}, z_o}$ was given and studied in [16, p. 210]. According to [16], the rational elliptic surface $S_{\mathcal{Q}, z_o}$ has a singular fibers of type I_4 and one singular fiber of type III. The Mordell-Weil lattice of S is $\text{MW}(S) \cong A_1^* \oplus A_3^*$ and the narrow Mordell-Weil lattice is $\text{MW}(S)^0 \cong A_1 \oplus A_3$.

Let $L_{x_o}, L_1, L_2, L_3, L_4$ be lines defined by

$$\begin{aligned} L_{x_o} : X = 0, \quad L_1 : 16T + X = 0, \quad L_2 : 15T + X = 0, \\ L_3 : 7T + X = 0, \quad L_4 : 12T + X = 0. \end{aligned}$$

These four lines give rise to the following dp-free sections,

$$\begin{aligned} s_{z_o,0} = (0, 4t^2), \quad s_{z_o,1} = (-16t, -48t), \quad s_{z_o,2} = (-15t, -t(t+45)), \\ s_{z_o,3} = (-7t, -3t(t-7)), \quad s_{z_o,4} = (-12t, 2t(t+18)), \end{aligned}$$

which are nothing but the basis given in [16]. We denote the elements corresponding to $s_{z_o,i}$ ($i = 0, \dots, 3$) by P_i ($i = 0, \dots, 3$), respectively. Note that the Gram matrix for P_0, P_1, P_2, P_3 is the first one in Example 3.3. We have

$$\begin{aligned} [2]P_0 = (x_1(t), y_1(t)) &= \left(\frac{1}{64}t^2 - \frac{41}{2}t + 315, -\frac{55}{512}t^2 + \frac{2637}{8}t - 5670 \right) \\ P_1 \dot{-} P_2 = (x_2(t), y_2(t)) &= (t^2 + 192t + 8640, -t^3 - 301t^2 - 27936t - 803520) \end{aligned}$$

Take $P = [2]P_0 = (x_1(t), y_1(t))$ and $r_{a,1}(t) = -t/8 + a$. By applying the method in Section 2.3, we have

$$\begin{aligned} &F(t, x, 1) - \{l_1(t, x)\}^2 \\ &= -\frac{1}{4096}(t^2 - 1312t - 64x + 20160) \\ &\quad (a^2t^2 - 1312a^2t - 64a^2x + 548at^2 + 16atx + 20160a^2 - 47232at + 9540t^2 \\ &\quad + 288tx + 64x^2 + 725760a - 425088t + 20736x + 6531840), \end{aligned}$$

where $l_1(t, x) = r_{a,1}(t)(x - x_1(t)) + y_1(t)$. Hence $C_{a,1} = C(r_{a,1}(t), [2]P_0)$ is a plane curve of degree 2 given as the zero locus of the second factor.

We next take $P = P_1 \dot{-} P_2 = (x_2(t), y_2(t))$ and $r_{b,2}(t) = -t + b$. By Section 2.3, we have

$$\begin{aligned} &F(t, x, 1) - \{l(t, x)\}^2 \\ &= -(t^2 + 192t - x + 8640)(b^2t^2 + 192b^2t - b^2x + 218bt^2 + 2btx + 8640b^2 + 38592bt \\ &\quad + 11865t^2 + 217tx + x^2 + 1607040b + 1928448t + 8649x + 74727360), \end{aligned}$$

where $l_2(t, x) = r_{b,2}(t)(x - x_2(t)) + y_2(t)$. Hence $C_{b,2} = C(r_{b,2}(t), P_1 \dot{-} P_2)$ is a plane curve of degree 2 given as the zero locus of the second factor.

By straightforward computation with computer similar to that in [6, Lemma 5], one can check that the three condition given in the beginning of this section are satisfied.

6 Proof of Theorem 0.5

In this section, we prove Theorem 0.5 by combining the result of Theorem 0.4 with a known Zariski triple constructed in [3].

Let \mathcal{Q} be the two nodal quartic given in Section 5. Assume that there exist 6 bisections D_1, \dots, D_6 of $\varphi_{\mathcal{Q}, z_o} : S_{\mathcal{Q}, z_o} \rightarrow \mathbb{P}^1$ as follows:

- (i) $s_{z_o}(D_i) = [2]s_{z_o, i}$ ($i = 1, 2$), $s_{z_o}(D_i) \in D_4^*$ ($i = 3, 4, 5, 6$).
- (ii) $\mathcal{C}_i = q \circ q_{z_o} \circ f_{\mathcal{Q}, z_o}(D_i)$ ($i = 1, \dots, 5$) are contact conics to \mathcal{Q} .
- (iii) \mathcal{C}_i is tangent to \mathcal{Q} at 4 distinct points for all i and \mathcal{C}_i intersects \mathcal{C}_j transversely if $i \neq j$.
- (iv) No three of \mathcal{C}_i 's meet at one point.
- (v) $\mathcal{C}_3, \mathcal{C}_4, \mathcal{C}_5, \mathcal{C}_6$ and \mathcal{Q} are the components of the Zariski triple given in [3, Theorem 1.4].

Put

$$\begin{aligned} \underline{\mathcal{C}}^{(1)} &= \mathcal{C}_1 + \mathcal{C}_2, & \underline{\mathcal{C}}^{(2)} &= \mathcal{C}_1 + \mathcal{C}_3, & \underline{\mathcal{C}}^{(3)} &= \mathcal{C}_3 + \mathcal{C}_4 \\ \underline{\mathcal{C}}^{(4)} &= \mathcal{C}_3 + \mathcal{C}_5, & \underline{\mathcal{C}}^{(5)} &= \mathcal{C}_3 + \mathcal{C}_6. \end{aligned}$$

By [3, Theorem 1.4], $(\mathcal{Q} + \underline{\mathcal{C}}^{(3)}, \mathcal{Q} + \underline{\mathcal{C}}^{(4)}, \mathcal{Q} + \underline{\mathcal{C}}^{(5)})$ is a Zariski triple. Note that this Zariski triple was distinguished by considering the “splitting type” defined for $(\mathcal{C}_i, \mathcal{C}_j; \mathcal{Q})$ which takes values $(0, 4), (1, 3), (2, 2)$ in this case. See [3] for details about splitting types.

Next, we consider $\sharp \left((\Phi_{\mathcal{Q}, \underline{\mathcal{C}}^{(i)}}^1)^{-1}(1) \right)$. We have

$$\sharp \left((\Phi_{\mathcal{Q}, \underline{\mathcal{C}}^{(i)}}^1)^{-1}(1) \right) = \begin{cases} 2 & i = 1 \\ 1 & i = 2 \\ 0 & i = 3, 4, 5 \end{cases}$$

By combining the two invariants, we have the following table where the top row indicates the value of $\sharp \left((\Phi_{\mathcal{Q}, \underline{\mathcal{C}}^{(i)}}^1)^{-1}(1) \right)$ and the left most column indicates the splitting type.

	0	1	2
(0, 4)	$\mathcal{Q} + \mathcal{C}_3 + \mathcal{C}_4$	—	$\mathcal{Q} + \mathcal{C}_1 + \mathcal{C}_2$
(1, 3)	$\mathcal{Q} + \mathcal{C}_3 + \mathcal{C}_5$	—	—
(2, 2)	$\mathcal{Q} + \mathcal{C}_3 + \mathcal{C}_6$	$\mathcal{Q} + \mathcal{C}_1 + \mathcal{C}_3$	—

Now it is immediate that $(\mathcal{Q} + \underline{\mathcal{C}}^{(1)}, \mathcal{Q} + \underline{\mathcal{C}}^{(2)}, \mathcal{Q} + \underline{\mathcal{C}}^{(3)}, \mathcal{Q} + \underline{\mathcal{C}}^{(4)}, \mathcal{Q} + \underline{\mathcal{C}}^{(5)})$ forms a Zariski 5-plet. Hence under the assumption that $\mathcal{C}_1, \dots, \mathcal{C}_6$ satisfying (i), ..., (v) exists, Theorem 0.5 is true.

Finally, we show that \mathcal{Q} and \mathcal{C}_i ($i = 1, \dots, 6$) as above exist by providing explicit equations. Let $\mathcal{C}_1, \dots, \mathcal{C}_6 \subset \mathbb{P}^2$ be conics given by $\mathcal{C}_1 = C(-\frac{1}{12}t, [2]s_0)$, $\mathcal{C}_2 = C(-\frac{1}{12}t + 1, [2]s_0)$, $\mathcal{C}_3 = C(\frac{1}{20}t, -s_2 + s_3)$, $\mathcal{C}_4 = C(\frac{1}{20}t + 1, -s_2 + s_3)$, $\mathcal{C}_5 = C(-\frac{1}{24}t, -s_1 + [2]s_2 - s_3 - s_4)$, $\mathcal{C}_6 = C(\frac{1}{6}t, [2]s_1 - s_2)$. The affine parts of $\mathcal{C}_1, \mathcal{C}_2$ are given by the following equations:

$$\begin{aligned} \mathcal{C}_1 : & \frac{174531500609375}{20736} - \frac{164065639375t}{10368} + \frac{86856575t^2}{20736} - \frac{33930625x}{144} + \frac{5825tx}{72} - x^2 = 0 \\ \mathcal{C}_2 : & \frac{173813141567975}{20736} - \frac{163641780439t}{10368} + \frac{86747399t^2}{20736} - \frac{33930481x}{144} + \frac{5813tx}{72} - x^2 = 0 \end{aligned}$$

As for the explicit equations of $\mathcal{C}_3, \dots, \mathcal{C}_6$, we refer to [3]. It can be easily checked that $\mathcal{C}_1, \dots, \mathcal{C}_6$ satisfy the conditions in Section 4, hence Theorem 0.5 is completely proved.

Remark 6.1 *As a final remark, we note that a Zariski 5-plet can be constructed in a similar way in the case where \mathcal{Q} is a quartic with one tacnode.*

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